Journal of Statistical Physics, Vol. 62, Nos. 1/2, 1991

On the Swendsen–Wang Dynamics. I. Exponential Convergence to Equilibrium

Fabio Martinelli,[1] Enzo Olivieri,[2] and Elisabetta Scoppola[3]

Received February 28, 1990; final July 26, 1990

We present rigorous results on the exponential convergence to equilibrium for the Swendsen–Wang stochastic dynamics for the d-dimensional Ising ferromagnet with external magnetic field h in the thermodynamic limit. We consider various situations, mainly in the low-temperature regime, in which boundary conditions are homogeneous and parallel or opposite to the external field. In the latter case we relate directly the tunneling from the metastable phase to the stable one with the exponential convergence to equilibrium.

KEY WORDS: Monte Carlo algorithms; Ising model; approach to equilibrium.

INTRODUCTION

In this paper we study the Swendsen–Wang[(1–3)] dynamics for the ferromagnetic Ising model and in particular its rate of convergence to equilibrium. The SW algorithm is reversible, i.e., it satisfies a detailed balance condition with respect to the Gibbs measure for the Ising model; it is used in Monte Carlo simulations because of its very rapid decay to equilibrium, this feature being particularly relevant near the critical point, where the critical slowing down is much less severe than, e.g., in Metropolis dynamics.

The SW algorithm is based on the Fortuin–Kasteleyn[(4,5)] representation of the Ising model and it has the advantage, with respect to the usual single-spin-flip Glauber dynamics, of updating in a very efficient way the configurations on large scales.

[1] Dipartimento di Matematica, Università "La Sapienza," Rome, Italy.

[2] Dipartimento di Matematica pura ed applicata, Università dell'Aquila, L'Aquila, Italy, CNR–GNFM.

[3] Dipartimento di Fisica, Università "La Sapienza," Rome, Italy.

The algorithm works as follows: starting from a configuration σ, we construct a new configuration σ' in two steps:

(i) First we construct the "bond configuration" $\{\gamma(b)\}$, $b=(x, x')$, $|x-x'|=1$, as follows: a bond (x, x') is defined to be "vacant," i.e., $\gamma((x, x'))=0$, if $\sigma(x)\neq\sigma(x')$; if $\sigma(x)=\sigma(x')$, then the bond (x, x') is defined to be "occupied," $\gamma(x, x')=1$, with probability $1-\exp(-\beta)$ and "vacant" with probability $\exp(-\beta)$, β being the inverse temperature.

(ii) Then, given $\{\gamma(b)\}$, we consider the connected sets of sites C, called "clusters," in the graph whose edges are the occupied bonds b. The second step consists in updating simultaneously all the spins in every cluster C. The updating is such that all the spins in C become parallel to the external magnetic field h with a probability $[1+\exp(-\beta\,|h|\,|C|)]^{-1}$ independently from cluster to cluster ($|C|$ denotes the cardinality of C).

We introduce homogeneous boundary conditions (b.c.) by imposing that the clusters which are connected to the boundary cannot flip and must preserve the same value of the spin as the boundary. A more detailed construction of the SW algorithm is given in Section 1.

In the present paper we shall mainly investigate the case when h is arbitrarily small but nonzero and β is consequently chosen large enough. The high-temperature case, which is relatively easy, also will be discussed for completeness.

One would like to be able to treat also the case in which $\beta>\beta_0$, β_0 a sufficiently large constant and h arbitrarily small; our techniques do not allow us, at the moment, to solve this case. The problem of convergence to equilibrium for single-spin-flip Glauber dynamics has been discussed in great detail by Holley,[6,7] but the analogous problem for the SW dynamics is new. In ref. 6, Holley considers general attractive finite-range stochastic Ising models; he reduces the proof of the exponential convergence to equilibrium to the validity of some mixing condition for the invariant Gibbs measure. This allows one to show exponential convergence for high enough temperature and/or high enough magnetic field.

The case of large β and small h is more interesting and difficult and, to our knowledge, is still open both for Glauber and SW dynamics. The reason for these difficulties is the same for both dynamics and it is related to the physical features of the equilibrium state: at low temperature and zero magnetic field the Ising model for $d\geqslant 2$ exhibits a phase transition and the configurations of the system have a kind of "symmetric double-well structure." For small h this feature is preserved even though now one of the two wells becomes deeper. It follows from this picture that to describe the global approach to equilibrium, uniformly in the initial configurations and in the boundary conditions, one needs to discuss the "tunneling" between the phases. It is clear that a rigorous description of this phenomenon entirely based on the dynamics it is of independent interest, and could be very useful in other fields, like simulated annealing.

In ref. 8 for the single-spin-flip Glauber dynamics and here for the SW dynamics we show exponential convergence in the above situation (h arbitrarily small and β very large, depending on h), i.e., we prove

$$|\mu_\Lambda(f(\sigma)) - Ef(\sigma_t)| \leqslant c_f \exp(-mt)$$

where $E(\cdot)$ denotes the expectation over the stochastic time evolution, μ_Λ is the invariant Gibbs measure in Λ, f is a function depending on finitely many spins, c_f is a positive constant depending only on f, and the positive constant m does not depend on the volume Λ.

More specifically, in Section 2, we consider the case of boundary conditions (b.c.) parallel to h where we are allowed to assume $\beta\,|h|$ very small but not too small ($\beta \gg 1$, $|h| \sim \exp -c\beta$). In Section 3 we consider the case of b.c. opposite to the field, where we assume $\beta\,|h|$ very large but $h \ll 1$. In Section 4 we consider the case of small β.

We remark here that, contrary to finite-range Glauber dynamics, for the SW algorithm, boundary conditions have *a priori* a big influence, since long-range interactions can arise from the presence of large clusters.

It turns out that when the b.c. are parallel to the magnetic field the structure of the configurations is that of a single well, i.e., there is only one "locally stable configuration" (i.e., a spin configuration that can be modified only with a probability going to zero as β tends to infinity). The reason is that large clusters opposite to the field are immediately flipped. However, when b.c. are opposite to h the double-well structure is effective; in fact, clusters connected to the boundary are pinned in the "wrong phase" and then if h is very small, there are two opposite locally stable configurations. Thus, final equilibrium can only be reached after the formation in the whole bulk of the "right phase." This phenomenon, which occurs via homogeneous nucleation, requires a much more detailed analysis and it is the subject of the second paper of this series.[9] After the appearance of the right phase the dynamics inside the bulk is practically the same as the one with b.c. parallel to h for time scales much larger than the typical time needed to reach equilibrium and we are back to the previous case.

Our approach is based on the proof of loss of memory of the initial conditions (in the language of Markov chains, it is a "coupling argument"). Roughly speaking, we show that two different initial spin configurations which evolve under the same noise (i.e., the random numbers involved in the definition of the dynamics are, at each step, the same for the evolution of both configurations) in a finite box of size L, after a time of order of $\log(L)$, become identical with a probability bounded below uniformly in L. It is easy to show that this property corresponds to the exponential convergence to the equilibrium measure. The analysis of this phenomenon has been initially developed by the authors in the case of other stochastic dynamics as diffusion processes given by small random perturbations of dynamical systems,[10-12] by means of a multiscale analysis of the time evolution of the process. Here the loss of memory is studied in terms of the Hamming distance (see Section 2) between two different initial data evolving under the same stochastic noise. The mechanism which is responsible for the loss of memory in the case of homogeneous boundary conditions parallel to the magnetic field h and large β is local in the following sense: in a short time scale both configurations will typically consist of a huge cluster of spins parallel to h attached to the boundary of the finite box and of small islands of opposite spins. These islands will rapidly flip in the direction of the external field and therefore they will become part of the huge cluster. Of course, due to thermal noise, other new islands will appear in the bulk of the sea of spins attached to the boundary. Since both configurations will have a very large portion of the cluster of the boundary in common, the newly formed islands will be identical for both, thus producing the loss of memory. In order to make the above picture rigorous, we need the external positive field h. However, contrary to single-spin-flip dynamics like the Metropolis or heath bath algorithms, we believe that the result should hold also without the external field (but keeping the homogeneous, e.g., $+$, boundary conditions). The reason is the following: for single-spin-flip dynamics there is a convincing argument by Huse and Fisher[15] (see also Sokal and Thomas[16]) predicting a nonexponential convergence to equilibrium, essentially based on the observation that large clusters of the wrong phase survive for a very long time (proportional to their area) under the dynamics. For the SW dynamics, however, big clusters of the wrong phase, which therefore are not attached to the boundary, can be flipped in a single move even without the external field. Thus, the role of the magnetic field here seems to us more technical than substantial.

Let us now sketch the idea of the proof. We adapt to our random dynamics an idea introduced by Von Dreyfus in his thesis[13] to analyze over different scales disordered systems and subsequently applied by Klein and Von Dreyfus to provide a simpler proof of Anderson localization for the Schrödinger equation.[14]

We define a sequence of length and time scales Λ_k and t_k and we consider the probability P_{k+1} of "conservation of memory" on scale $k+1$:

$$P_{k+1} = P(\exists x \in \Lambda_{k+1}\colon \Phi_{t_{k+1}}^{\Lambda_{k+1}}(\sigma)(x) \neq \Phi_{t_{k+1}}^{\Lambda_{k+1}}(\eta)(x))$$

where $\Phi_t^{\Lambda}(\sigma)$ is the evolution at time t of the initial configuration σ in a box Λ. We estimate P_{k+1} in terms of the probability P_k of the same event on scale k and we get

$$P_{k+1} \leqslant a_k P_k^2 + b_k, \qquad \forall k \geqslant k_0$$

It is not difficult to see that for suitable a_k and b_k, P_k becomes exponentially small for large k provided P_{k_0} is sufficiently small.

In this way we are reduced to a problem on a fixed finite scale.

In the case of SW dynamics the finite-size condition turns out to be very easy to verify in the range of parameters under consideration.

1. DEFINITIONS AND NOTATION

We start by constructing the dynamics with + boundary conditions and uniform magnetic field h. We first introduce the notation.

(i) Λ will denote a generic finite subset of Z^d, Λ_L^x, $x \in Z^d$, $L \in N$, will denote the cube in Z^d of side L, L odd, centered at x.

(ii) The unordered pair b in Z^d: $b = (x, y)$, $|x - y| = 1$, is called a bond Λ; Λ^* is the set of all bonds (x, y) such that either x or y or both belong to Λ.

(iii) $\sigma \in \{-1, 1\}^{|\Lambda_L^x|}$ denotes a generic configuration of plus or minus spins in Λ_L^x.

(iv) $\mathscr{C}_{\Lambda_L^x}$ is the family of all "geometric clusters" C in $\bar{\Lambda} = \{x; \exists b \in \Lambda^*; x \in b\}$. A geometric cluster C is a subset of Z^d which is connected in the following sense: $\forall x$, $y \in C$ then exists a chain of nearest neighbor sites in C connecting x to y:

$$x_1 \cdots x_n\colon\ x_1 = x, \quad x_n = y, \quad |x_{i+1} - x_i| = 1, \quad i = 1,..., n-1$$

Now, given Λ, let v_b be numbers in $\{0, 1\}$ associated to each bond $b \in \Lambda^*$ and let ξ_C be numbers in $[0, 1]$ associated to each geometric cluster $C \in \mathscr{C}_\Lambda$, respectively.

Given the numbers v_b and ξ_C, we construct out of a configuration σ a new configuration σ' as follows. From σ we first generate a new configuration γ of occupied $[\gamma(b) = 1]$ and vacant $[\gamma(b) = 0]$ bonds, by setting

$$\gamma(b) = \frac{1 + \sigma_b}{2} v_b$$

where $\sigma_b = \sigma_x \sigma_y$ if $b = (x, y)$. The configuration γ can be identified as the subset of the occupied bonds in Λ^*. Sometimes, in order to denote the configuration (and the corresponding subset of Λ^*) γ obtained starting from σ, we use the symbol γ_σ (γ_σ depends of course on the numbers ν_b). We will say that two n.n. sites (x, x') are connected in the bond configuration γ if $\gamma(x, x')=1$, i.e., the bond (x, x') is occupied in γ. The maximal connected components C (with respect to the configuration γ) are called "γ-clusters" or more simply clusters. They are of course in particular geometric clusters and may reduce to a single site.

For a geometric cluster C which is also a γ-cluster we often write $C \subset \gamma_\sigma$. Now, for any $C \subset \gamma_\sigma$ we set

$$\begin{aligned} &\sigma'(x)=1 \qquad \forall x \in C \quad \text{if either} \quad \xi_C < (1+e^{-\beta h |C|})^{-1} \quad \text{or} \quad C \cap \partial\Lambda \neq \varnothing \\ &\sigma'(x)=-1 \quad \forall x \in C \quad \text{if} \quad \xi_C \geqslant (1+e^{-\beta |C|})^{-1} \quad \text{and} \quad C \cap \partial\Lambda = \varnothing \end{aligned} \tag{1.1}$$

where $|C| \equiv \#\{x \in C\}$ and $\partial\Lambda = \{x \notin \Lambda \colon |x-y| = 1\}$.

Let us now consider two sequences of numbers,

$$\omega \equiv (\{\nu_b(t)\}_{\substack{t \in N \\ b \in \Lambda^*}}, \{\xi_C(t)\}_{\substack{t \in N \\ C \in \mathscr{C}_\Lambda}})$$

that we think of as the realization of two mutually independent processes with values in $\{0, 1\}$ and $[0, 1]$, respectively, each of which is a collection of independent identically distributed random variables (iid rv) with distribution:

$$\begin{aligned} &\nu_b = 0 \quad \text{with probability } \exp(-\beta) \\ &\nu_b = 1 \quad \text{with probability } 1-\exp(-\beta) \end{aligned}$$

and uniform distribution in $(0, 1)$ for the ξ_C.

Given ω, we finally construct a random flow on $\{-1, 1\}^\Lambda$, $\{\phi_t^{\Lambda,\omega}(\cdot)\}_{t \in N}$ by applying at each time step t the rule (1.1) with numbers $\nu_b(t)$, $\xi_C(t)$. Sometimes, for notational convenience, we will write

$$\sigma_t^\omega(x) = \phi_t^{\Lambda,\omega}(\sigma)(x) \tag{1.2}$$

Remarks. (i) The boundary condition $+1$ at the boundary of Λ is taken into account by the condition that any cluster C touching $\partial\Lambda$ is set equal to $+1$. Other boundary conditions may be considered, e.g., periodic or open.

(ii) The case of a nonconstant magnetic field can also be discussed; in this case $h|C|$ is replaced by $\sum_{x \in C} h_x$ in (1.1).

(iii) Notice that if $\Lambda' \subset \Lambda$, then one can compare the random flows $\phi_t^{\Lambda,\omega}$, $\phi_t^{\Lambda',\omega}$ as follows: given σ in Λ, one constructs $\hat\sigma$ in Λ' by the rule

$$\begin{aligned} &\hat\sigma(x) = \sigma(x) \qquad \text{if} \quad x \in \Lambda' \backslash \partial\Lambda' \\ &\hat\sigma(x) = +1 \qquad \text{if} \quad x \in \partial\Lambda' \end{aligned}$$

The evolutions $\phi_t^{\Lambda,\omega}(\sigma)$ and $\phi_t^{\Lambda',\omega}(\hat{\sigma})$ are constructed by means of the same random numbers $(\nu_b(t), \xi_C(t))$ if b and C are in Λ'. However, a cluster C intersecting $\partial\Lambda'$ is set equal to $+1$ for the dynamics $\phi_t^{\Lambda',\omega}$, but may be -1 for the dynamics $\phi_t^{\Lambda,\omega}$. This observation will be exploited in a crucial way in the next section.

The above-defined dynamics satisfies the detailed balance condition for the Hamiltonian defined below, which in turn shows that the only invariant measure for $\phi_t^{\Lambda,\omega}$ is the Gibbs state on Λ, with $+$ boundary conditions and magnetic field h, at inverse temperature β. Let

$$H^\Lambda(\sigma) = -\frac{1}{2}\sum_{b\in\Lambda^*} n_b(\sigma) - \frac{h}{2}\sum_{x\in\Lambda}\sigma(x) - \frac{1}{2}\sum_{\substack{x\in\Lambda;\\ \mathrm{dist}(x,\Lambda^c)=1}} \sigma(x) f(x) \tag{1.3}$$

where $f(x) = \natural\{y\notin\Lambda: |x-y|=1\}$, and let $P(\sigma\to\sigma')\equiv \mathrm{Prob}(\phi_1^{\Lambda,\omega}(\sigma)=\sigma')$. Then we have

$$\frac{P(\sigma\to\sigma')}{P(\sigma'\to\sigma)} = \exp\{-\beta[H^\Lambda(\sigma') - H^\Lambda(\sigma)]\} \tag{1.4}$$

The proof of (1.4) is given, for completeness, in the Appendix.

Warning. Times and length scales are always integer; however, for simplicity we always omit the integer part symbol $[\cdot]$.

2. APPROACH TO EQUILIBRIUM IN THE CASE $h > 0$ AND $\beta \gg 1$

In this section we study the rate of convergence to equilibrium for the dynamics discussed in Section 1 with magnetic field parallel to the boundary condition and $\beta \gg 1$. This case is usually considered quite complicated, since the spins $\sigma_t(x)$ are strongly correlated, contrary to the case $\beta \ll 1$, where they almost behave like i.i.d. random variables. Here we do not need this restriction; actually, we will consider h as small as $e^{-c\beta}$ for suitable $c > 0$, but $h > 0$ will be crucial. The case of magnetic field opposite to the boundary condition requires a detailed discussion of metastability and it is postponed to the next section.

Our point of view for studying the approach to the equilibrium is to analyze the time behavior of the Hamming distance

$$\rho(\sigma_t, \eta_t) \equiv \frac{1}{4}\sum_x [\sigma_t(x) - \eta_t(x)]^2 \tag{2.1}$$

between two configurations σ_t, η_t evolving with the same random flow $\phi_t^{\Lambda,\omega}$. We will prove that if the time t is taken large enough, depending on the size of the box Λ, then with very large probability, $\rho(\sigma_t, \eta_t) = 0$. With this result we will prove exponential approach to equilibrium.

For $k \in N$ let $L_k = 10^k$ and $t_k = 2^k$ be a sequence of lengths and time scales. We set $\Lambda_k \equiv \Lambda^0_{L_k}$ and

$$P_k \equiv \sup_{\sigma, \eta \in \{-1,1\}^{\Lambda_k}} P(\rho(\phi^{\Lambda_k}_{t_k}(\sigma), \phi^{\Lambda_k}_{t_k}(\eta)) > 0) \tag{2.2}$$

We will prove the following result.

Theorem 1. There exist positive constants h_0, β_0, a depending on d, such that if $\beta > \beta_0$, $h > \exp(-h_0\beta)$, there exists $m(\beta, h) > 0$ such that for any $k > a\beta$

$$P_k \leqslant e^{-mt_k}$$

An easy consequence of the above result is the exponential approach to equilibrium. Let f be a real function on $\{-1, 1\}^{Z^d}$ of compact support S_f, i.e., $f(\sigma) \equiv f(\{\sigma(x)\}_{x \in S_f})$, with $|S_f| < \infty$, and let

$$\mu_k(f) = \frac{1}{Z} \sum_{\sigma \in \{-1,1\}^{\Lambda_k}} f(\sigma) \exp\{-\beta H^{\Lambda_k}(\sigma)\} \tag{2.3}$$

with

$$\begin{aligned} \mu_k &\equiv \mu_{\Lambda_k} \\ Z &= \sum_{\sigma \in \{-1,1\}^{\Lambda_k}} \exp\{-\beta H^{\Lambda_k}(\sigma)\} \end{aligned} \tag{2.4}$$

and $H^{\Lambda_k}(\sigma)$ as in Section 1.

In the same hypotheses of Theorem 1, we have the following result.

Corollary. There exist constants $C_f > 0$ and $m = m(\beta) > 0$ such that

$$\sup_\sigma |\mu_k(f) - Ef(\phi^{\Lambda_k}_t(\sigma))| < C_f \exp(-mt)$$

uniformly in k.

As is clear from the proofs of the above results, it will be crucial to compare dynamics on different length scales and to prove that up to a certain time scale, they are indistinguishable. This is the content of the next lemma.

Lemma 1. Let $h > 0$. For any $k' < k$ and $\sigma \in \{-1, 1\}^{\Lambda_k}$, let $\mathscr{D}^k_{k'}$ be the event

$$\{\exists t \leqslant 2t_{k'}, \exists x \in \Lambda^0_{L_k - L_{k'}};\ \phi^{\Lambda^x_{k'}}_t(\sigma)(y) \neq \phi^{\Lambda_k}_t(\sigma)(y)$$
$$\text{for some } y \in \Lambda^x_{k'} \text{ with } \operatorname{dist}(y, \partial\Lambda^x_{k'} \backslash \partial\Lambda_k) \geqslant L_{k'}/2\}$$

Then, if

$$4t_{k'}(L_k - L_{k'})^d L_{k'}^d \exp\left\{-\frac{\beta h}{2}\left(\frac{L_{k'}}{2}\right)^{1/2}\right\} < 1$$

we have

$$P(\mathscr{D}_{k'}^k) \leqslant \exp\left\{-\frac{\beta h}{2}\left(\frac{L_{k'}}{2}\right)^{1/2}\right\}$$

Proof of Lemma 1. Given $\sigma \in \{-1, 1\}^{\Lambda_k}$, let $\Omega_{k'}^k$ be the event

$$\begin{aligned}\Omega_{k'}^k \equiv \{&\exists s \in [0, 2t_{k'}], \exists x \in \Lambda_{L_k - L_{k'}}^0; \exists C \subset \gamma_{\phi_s^{\Lambda_k}(\sigma)} \cup \gamma_{\phi_s^{\Lambda_{k'}^x}(\sigma)} \\ &\text{with diam } C > (L_{k'}/2)^{1/2}; C \cap \Lambda_{k'}^x \neq \varnothing \text{ and } \xi_C^{(s)} > (1 + e^{-\beta h |C|})^{-1}\}\end{aligned} \tag{2.5}$$

where diam C denotes the diameter of the cluster C: diam $C = \sup_{x, y \in C} |x - y|$. The probability of $\Omega_{k'}^k$ is estimated uniformly in σ and k by

$$\begin{aligned}P(\Omega_{k'}^k) &\leqslant \sup_{\sigma} 2t_{k'} \sup_{s \in [0, 2t_{k'}]} (L_k - L_{k'})^d \\ &\quad\times \sup_{x \in \Lambda_{L_k - L_{k'}}^0} \sum_{\substack{C \cap \Lambda_{k'}^x \neq \varnothing \\ \text{diam } C > (L_{k'}/2)^{1/2}}} e^{-\beta h |C|} \cdot P(C \subset \gamma_{\phi_s^{\Lambda_k}(\sigma)} \cup \gamma_{\phi_s^{\Lambda_{k'}^x}(\sigma)}) \\ &\leqslant 2t_{k'}(L_k - L_{k'})^d \sup_{\sigma} \sup_{s \in [0, 2t_{k'}]} \\ &\quad\times \sup_{x \in \Lambda_{L_k - L_{k'}}^0} \sum_{y \in \Lambda_{k'}^x} \sum_{\substack{C; \\ y \in C \\ \text{diam } C > (L_{k'}/2)^{1/2}}} P(C \subset \gamma_{\phi_s^{\Lambda_k}(\sigma)} \cup \gamma_{\phi_s^{\Lambda_{k'}^x}(\sigma)}) \, e^{-\beta h |C|} \\ &\leqslant 4t_{k'}(L_k - L_{k'})^d L_{k'}^d \exp[-\beta h (L_{k'}/2)^{1/2}]\end{aligned} \tag{2.6}$$

since

$$\sum_{\substack{C; \\ y \in C \\ \text{diam } C > (L_{k'}/2)^{1/2}}} P(C \subset \gamma_{\phi_s^{\Lambda_k}(\sigma)} \cup \gamma_{\phi_s^{\Lambda_{k'}^x}(\sigma)}) \leqslant 2$$

Next we show that

$$\mathscr{D}_{k'}^k \quad \text{implies} \quad \Omega_{k'}^k \tag{2.7}$$

Let

$$\begin{aligned}\mathscr{F}_s \equiv \{&\phi_s^{\Lambda_{k'}^x}(\sigma)(y) = \phi_s^{\Lambda_k}(\sigma)(y) \; \forall x \in \Lambda_{L_k - L_{k'}}^0, \forall y \in \Lambda_{k'}^x \\ &\text{with dist}(y, \partial\Lambda_{k'}^x \backslash \partial\Lambda_k) > 2s(L_{k'}/2)^{1/2}\}\end{aligned} \tag{2.8}$$

We proceed by induction by proving that $(\Omega^k_{k'})^c$ implies $\mathscr{F}_1 \cap \mathscr{F}_2 \cap \cdots \cap \mathscr{F}_{2t_{k'}} \subseteq (\mathscr{D}^k_{k'})^c$. In fact, if $(\Omega^k_{k'})^c$ holds and $\mathscr{F}_s$ is true, then $\mathscr{F}_{s+1}$ is true. Suppose $\mathscr{F}_{s+1}$ false; then there exist a site $x \in \Lambda^0_{L_k - L_{k'}}$ and a site $x_0 \in \Lambda^x_{k'}$ with $\operatorname{dist}(x_0, \partial\Lambda^x_{k'} \backslash \partial\Lambda_k) > 2(s+1)(L_{k'}/2)^{1/2}$ with

$$\phi^{\Lambda^x_{k'}}_{s+1}(\sigma)(x_0) \neq \phi^{\Lambda_k}_{s+1}(\sigma)(x_0)$$

e.g.,

$$\phi^{\Lambda^x_{k'}}_{s+1}(\sigma)(x_0) = -1, \qquad \phi^{\Lambda_k}_{s+1}(\sigma)(x_0) = 1$$

Thus, by (1.1), x_0 belonged at time s to two different clusters C, C' with

$$C \subset \gamma_{\phi^{\Lambda_k}_s(\sigma)} \qquad \text{and} \qquad C' \subset \gamma_{\phi^{\Lambda^x_{k'}}_s(\sigma)}$$

By $\mathscr{F}_s$ we have that $\operatorname{diam} C' > (L_{k'}/2)^{1/2}$, otherwise C would have been equal to C'. But $\operatorname{diam} C' > (L_{k'}/2)^{1/2}$ is impossible by $(\Omega^k_{k'})^c$.

In conclusion, $(\Omega^k_{k'})^c$ and $\mathscr{F}_0$ imply $\mathscr{F}_1 \cap \cdots \cap \mathscr{F}_{2t_{k'}}$ and thus (2.7) is proved. By (2.6) the lemma is proved, provided k' is sufficiently large, that is,

$$4t_{k'}(L_k - L_{k'})^d L^d_{k'} \exp[-\beta h(L_{k'}/2)^{1/2}] \leqslant \exp[-(\beta h/2)(L_{k'}/2)^{1/2}]$$

Proof of Theorem 1. We will establish the following recursion relation

Lemma 2. The relation

$$P_{k+1} \leqslant L^d_{k+1} P^2_k + 2\exp[(-\beta h/2)(L_k/2)^{1/2}] \tag{2.9}$$

holds for k large enough.

Proof. We start from

$$P_{k+1} \leqslant 2 \sup_\sigma P(\exists x \in \Lambda^0_{L_k - L_{k'}};\ \phi^{\Lambda^x_k}_{t_{k+1}}(\sigma)(x) \neq \Lambda^{\Lambda_{k+1}}_{t_{k+1}}(\sigma)(x))$$
$$+ \sup_{\sigma,\eta} P(\exists x \in \Lambda^0_{L_k - L_{k'}};\ \phi^{\Lambda^x_k}_{t_{k+1}}(\sigma) \neq \phi^{\Lambda^x_k}_{t_{k+1}}(\eta)) \tag{2.10}$$

By using Lemma 1 and the definition of t_k, the first term in the rhs of (2.10) is estimated by

$$2\exp\{-(\beta h/2)(L_k/2)^{1/2}\} \tag{2.11}$$

if

$$4t_k(L_{k+1} - L_k)^d L^d_k \exp\{-(\beta h/2)(L_k/2)^{1/2}\} \leqslant 1 \tag{2.12}$$

The second term, by using the Markov property and $t_{k+1}=2t_k$, is estimated by

$$L_{k+1}^d P_k^2 \tag{2.13}$$

Thus, (2.9) and the lemma are proved for k sufficiently large.

Let $k_0(\beta)=a\beta$, $a>0$, and let h_0 be smaller than $\frac{1}{2}a\ln(\sqrt{10})$. Then, if $h>e^{-\beta h_0}$ and if for any d we choose β large enough, (2.12) holds $\forall k\geqslant k_0$.

Next we set $f_k\equiv P_k 11^{d_k}$, and from (2.9) we easily get

$$f_{k+1}\leqslant f_k^2+e^{-(\beta h/4)\,t_k} \tag{2.14}$$

$\forall k>k_0$ and β sufficiently large.

We know that (see Lemma 2.1 in ref. 6)

$$f_k\leqslant \tfrac{1}{2}[2\cdot(f_{k_0}\vee e^{-(\beta h/8)\,t_{k_0}})]^{2^{k-k_0}} \tag{2.15}$$

Thus the theorem follows if we can show that

$$2f_{k_0}<1 \tag{2.16}$$

provided β_0 is large enough.

To verify (2.16), we first remark that

$$\begin{aligned} &P(\forall s\in[0,t_{k_0}] \text{ and } \forall\text{bond } b\in\Lambda_{k_0}^*,\ v_b(s)=1)\\ &\quad\geqslant 1-t_{k_0}(L_{k_0})^d e^{-\beta}>1-e^{-\beta/2} \end{aligned} \tag{2.17}$$

provided a is such that

$$2^{a\beta}10^{a\beta d}d<e^{\beta/2} \tag{2.18}$$

Thus we can estimate f_{k_0} as

$$\begin{aligned} f_{k_0}\leqslant{}& 11^{da\beta}e^{-\beta/2}+11^{da\beta}\sup_{\sigma,\eta}P(\forall s\in[0,t_{k_0}],\ \forall b\in\Lambda_{k_0}^*,\ v_b(s)=1\\ &\text{and } \rho(\phi_{t_{k_0}}^{\Lambda_{k_0}}(\sigma),\phi_{t_{k_0}}^{\Lambda_{k_0}}(\eta))>0) \end{aligned} \tag{2.19}$$

The probability appearing in the second term in the rhs of (2.19) is in turn estimated by

$$2\sup_{\sigma}P(\forall s\in[0,t_{k_0}],\ \forall b\in\Lambda_{k_0}^*,\ v_b(s)=1;\ \phi_{t_{k_0}}^{\Lambda_{k_0}}(\sigma)(x)=-1 \text{ for some } x) \tag{2.20}$$

In order to estimate (2.20), let us introduce the following random variable:

$$\psi(t,x)=\min_{\substack{\{x_0,x_1\cdots x_n\}\\ x_0=x\\ x_n\in\partial\Lambda_{k_0}\\ |x_i-x_{i-1}|=1}}\sum_{i=1}^{n}\frac{1-\Phi_t^{\Lambda_{k_0}}(\sigma)(x_i)\,\Phi_t^{\Lambda_{k_0}}(\sigma)(x_{i-1})}{2} \tag{2.21}$$

The variable $\psi(t, x)$ counts the number of Peierls contours that one has to cross starting from x before reaching $\partial\Lambda_{k_0}$. Thus, if $\psi(t, x) = 0\ \forall x$, then $\phi_t^{\Lambda_{k_0}}(\sigma)(x) = 1\ \forall x$. Now, if for any $s \in [0, t_{k_0}]$ and any bond $b \subset \Lambda_{k_0}$, $v_b(s) > e^{-\beta}$, then the variable $\psi(t, x)$ cannot increase as t varies and if $\langle \cdot \rangle$ denotes the average over the process $\{\xi_C(s)\}_{C \in \mathscr{C}_{\Lambda_{k_0}}, s \in N}$ then one easily gets

$$\langle \psi(t+1, x) \rangle \leqslant \tfrac{3}{4} \langle \psi(t, x) \rangle \tag{2.22}$$

$\forall h > 0$. By using the Chebyshev inequality, we can conclude

$$\begin{aligned} &P(\forall s \in [0, t_{k_0}], \forall b \in \Lambda_{k_0}^*, v_b(s) = 1; \psi(t_{k_0}, x) \geqslant 1 \text{ for some } x) \\ &\quad \leqslant L_{k_0}^d L_{k_0} (\tfrac{3}{4})^{t_{k_0}} \end{aligned} \tag{2.23}$$

Using finally (2.23) as an estimate of (2.20), we get that the rhs of (2.19) is bounded by

$$f_{k_0} \leqslant 11^{da\beta} e^{-\beta/2} + 11^{da\beta} \cdot 10^{a\beta[d+1]} (\tfrac{3}{4})^{2^{a\beta}} \leqslant \tfrac{1}{2}$$

provided β is large enough and a is chosen so that both (2.18) and $11^{da\beta} e^{-\beta/2} \leqslant e^{-\beta/4}$ hold true. The proof of the theorem is complete.

Proof of the Corollary. Let t be so large that $\log t/\log 2 > a\beta$ and take $k_t = \log t/\log 2$ (with a as in Theorem 1). If t was taken large enough, then clearly

$$\operatorname{dist}(\partial\Lambda_{k_t}, S_f) \geqslant \tfrac{1}{4} L_{k_t} \tag{2.24}$$

so that, using again Lemma 1 and the definition of k_t, $\forall k > k_t$,

$$\sup_\sigma P(\exists x \in S_f; \phi_t^{\Lambda_{k_t}}(\sigma)(x) \neq \phi_t^{\Lambda_k}(\sigma)(x)) \leqslant \exp\{-(\beta h/2)(L_{k_t}/2)^{1/2}\} \tag{2.25}$$

if $4tL_{k_t}^d \exp\{-(\beta h/2)(L_{k_t}/2)^{1/2}\} \leqslant 1$, which is certainly true for β sufficiently large. Thus,

$$\begin{aligned} &\sup_\sigma |\mu_k(f) - Ef(\phi_t^{\Lambda_k}(\sigma))| \\ &\quad = \sup_\sigma \left| \sum_\eta \frac{1}{Z} e^{-(\beta/2) H^{\Lambda_k}(\eta)} [Ef(\phi_t^{\Lambda_k}(\eta)) - Ef(\phi_t^{\Lambda_k}(\sigma))] \right| \\ &\quad \leqslant 2e^{-(\beta h/2)t} \sup_\sigma |f(\sigma)| + \sup_{\sigma,\eta} |Ef(\phi_t^{\Lambda_{k_t}}(\eta)) - Ef(\phi_t^{\Lambda_{k_t}}(\sigma))| \\ &\quad \leqslant (2e^{-(\beta h/2)t} + e^{-mt}) \sup_\sigma |f(\sigma)| \end{aligned} \tag{2.26}$$

if $k > k_t$.

In the first equality in (2.26) we used the invariance of the measure $\mu_k(\cdot)$ with respect to the random flow $\phi_t^{\Lambda_k}(\cdot)$. It is clear that (2.26) proves the corollary.

3. APPROACH TO EQUILIBRIUM IN THE CASE $h<0$, + BOUNDARY CONDITIONS, AND β LARGE

In this section we analyze the more difficult case of uniform magnetic field opposite to the boundary conditions. We restrict ourselves to $0<|h|\ll 1$ but $\beta\,|h|\gg 1$ and also to $d=2$, the one-dimensional case being less interesting.

When the magnetic field is opposite to the boundary conditions the problem of the exponential approach to equilibrium, represented by the minus phase, cannot be attacked directly as in Section 2. The technical reason is that Lemma 1 fails, at least as stated.

In fact, this is suggested by the following remark: when $\phi_t^\Lambda(\phi)$ has reached the "− phase" its dynamics in the bulk of Λ should be very close to a dynamics on a smaller box $\Lambda'\subset\Lambda$ with − boundary conditions and not with + b.c. as in Lemma 1; while before reaching the "− phase" the + boundary conditions on $\partial\Lambda'$ should be more appropriate. Thus, one has to show that any initial configuration σ after a certain typical time $t(\beta, h)$ reaches the − phase and then try to compare its dynamics in the bulk of Λ with the dynamics restricted to a smaller box $\Lambda'\subset\Lambda$ with boundary conditions *parallel* to the magnetic field, for which the results of Section 2 apply.

In this paper we limit ourselves to the analysis of the second part of the above strategy and we use, without proof, the results on the transition from the + phase to the − phase proved in ref. 9. Let

$$t(\beta, h)=\exp\left(\frac{4\beta}{|h|}\right) \qquad \text{if} \quad d=2 \tag{3.1}$$

Note that $t(\beta, h)$ may be written as

$$t(\beta, h)=\exp(\beta\,\Delta H) \tag{3.2}$$

where ΔH is the difference in energy between a configuration $\hat\sigma$: $\hat\sigma(x)=1$ if $x\notin\Lambda_{l_c}^0$ and $\hat\sigma=-1$ if $x\in\Lambda_{l_c}^0$ and the + configuration where $l_c\simeq 2/h$. The critical size l_c is found statically by imposing that ΔH attains its maximum as a function of l and dynamically by imposing that the droplet $\Lambda_{l_c}^0$ of -1 in the sea of $+1$ is more likely to grow than to shrink.

In ref. 9 the following fundamental estimate has been proved.

Theorem 2. $\exists c>0$ such that given $h<0$, sufficiently small in absolute value, there exist constants $\beta_0(h)$, $L_0(h)$ such that for any set $\Lambda \subset \Lambda_L$ with $\text{dist}(\Lambda, \partial\Lambda_L) > L_0(h)$:

$$P(\exists\sigma;\, \phi_t^{\Lambda_L}(\sigma)(x) = 1 \quad \forall x \in \Lambda) \leqslant e^{-k(\beta)\,|\Lambda|}$$

where $k(\beta) \nearrow +\infty$ as $\beta \nearrow +\infty$, provided

$$\beta > \beta_0, \qquad L > L_0, \qquad t > t(\beta, h)\, e^{c\beta}$$

Using this result, it is simple to prove, by means of the Peierls argument, that for $t > t(\beta, h, c) \equiv t(\beta, h)\, e^{c\beta}$ and n sufficiently large,

$$\begin{aligned}
&\sup_\sigma P(\exists s \in [t(\beta, h, c), T] \text{ and a nearest neighbor self-avoiding path}\\
&\qquad \Gamma = \{x_1,..., x_n\} \text{ with } \phi_s^\Lambda(\sigma)(x_i) = +1,\ i = 1,..., n \text{ and } \Gamma \cap \partial\Lambda_L = x_1)\\
&\quad \leqslant T(2L)\, e^{-k(\beta)n/2}
\end{aligned} \tag{3.3}$$

provided $\text{dist}(\{x_{n/2},..., x_n\}, \partial\Lambda) \geqslant L_0(h)$ with

$$k(\beta) \nearrow +\infty \qquad \text{as} \quad \beta \to +\infty$$

We take $n = T = \log L$ with $\log L \gg e^{\beta/h}$. Thus, with probability larger than $1 - \exp[-\frac{1}{2}k(\beta) \log L]$ the $+$ cluster of the boundary does not reach the box $\Lambda_{L'}^0$ with $L' = L - \log L$.

Next we show that the dynamics inside $\Lambda'' \equiv \Lambda_{L'/2}^0$ after a time $t(\beta, h, c)$ is very well approximated by the dynamics $\phi_t^{\Lambda'', -}(\cdot)$ with minus boundary conditions on $\partial\Lambda''$. We have

$$\begin{aligned}
&\sup_\sigma P(\exists x \in \Lambda'';\, \exists t \in [t(\beta, h, c), T],\, \phi_t^{\Lambda,+}(\sigma)(x) \neq \phi_{t-t(\beta,h,c)}^{\Lambda'',-}(\phi_{t(\beta,h,c)}^{\Lambda,+}(\sigma))(x))\\
&\qquad \leqslant \exp[-\tfrac{1}{4}k(\beta) \log L]
\end{aligned} \tag{3.4}$$

The proof is omitted since it is almost the same as the one given in Section 2. The main ingredients are:

(i) Big clusters in $\gamma_{\phi_t^\Lambda(\sigma)}$ or in $\gamma_{\phi_{t-t(\beta,h,c)}^{\Lambda''}(\phi_{t(\beta,h,c)}^\Lambda(\sigma))}$ which *do not* touch the boundary $\partial\Lambda$ behave similarly, i.e., they become -1 with high probability.

(ii) Because of (3.3), there exists with large probability no effective cluster for the dynamics which touches $\partial\Lambda_L$ and $\partial\Lambda''$. These clusters are the only ones that may detect the difference in the boundary conditions in $\phi_t^\Lambda(\cdot)$ and $\phi_t^{\Lambda''}(\cdot)$. Once (3.4) has been established, the analog of the corollary becomes trivial.

Proposition 1. Let $f\colon \{-1, 1\}^{\Lambda_{L/2}} \to R$ be of compact support. Let $h<0$ be sufficiently small in absolute value. Then there exist constants C_f, $\beta_0(h)$, $m>0$ such that, $\forall \beta > \beta_0(h)$,

$$\sup_\sigma |\mu_{\Lambda_L}(f) - Ef(\phi_t^{\Lambda_L}(\sigma))| \leqslant C_f e^{-m(t - t(\beta, h, c))}$$

4. APPROACH TO EQUILIBRIUM. THE HIGH-TEMPERATURE CASE $\beta \ll 1$

We conclude this work with a short discussion of the high-temperature $\beta \ll 1$ case. We do this just for completeness, since, as in Glauber dynamics, the high-temperature regime is rather simple as long as one is sufficiently far away from the critical point. From a qualitative point of view in the high-temperature regime bonds are cut $[n'_b = -1]$ with very large probability and therefore two arbitrary configurations σ and η become identical in a short time.

In order to prove the exponential approach to equilibrium, it is sufficient to prove the high-temperature analog of Theorem 1, and in turn this follows if: (a) one controls the range of information transmission (see, e.g., Lemma 1); (b) there exists a scale k_0 such that

$$P_{k_0} \leqslant \tfrac{1}{2} 11^{-dk_0}$$

Condition (a) is achieved through the following lemma.

Lemma 3. There exists $\beta_0 > 0$ such that if $\beta < \beta_0$, there exists $k(\beta)$ with $k(\beta) \nearrow +\infty$ as $\beta \searrow 0$ such that

$$\sup_\sigma P(\exists s \in [0, t]; \exists C \subset \gamma_{\phi_s^\Lambda(\sigma)} \text{ with } C \cap \Lambda_{t^3} \neq \varnothing, \operatorname{diam} C > t) \leqslant e^{-k(\beta) t}$$

In the above estimate L is of course required to be greater than const $\cdot\, t^3$.

Proof. The probability appearing in the lemma can be estimated by

$$\begin{aligned} t(t^3)^d \sup_{x \in \Lambda_{t^3}} \sup_{\sigma, s \leqslant t} &P(\exists \text{ a nearest neighbor path in } \gamma_{\phi_s^\Lambda(\sigma)}\ x_1, \ldots, x_n, n > t \\ &\quad \text{with } |x_i - x_{i-1}| = 1 \text{ and } \gamma(b_i) = 1\ \ \forall b_i = (x_i, x_{i+1}) \text{ and } x_1 = x) \\ &\leqslant t(t^3)^d \sum_{n \geqslant t} (1 - e^{-\beta})^n (2d)^n \leqslant e^{-k(\beta) t} \end{aligned} \tag{4.1}$$

for $\beta \ll 1$.

With the help of the above lemma, one easily proves the analog of Lemma 1:

$$\sup_{\sigma} P(\phi_{t_{k+1}}^{\Lambda_{k+1}}(\sigma)(0) \neq \phi_{t_{k+1}}^{\Lambda_k}(\sigma)(0)) \leqslant e^{-m(\beta)\, t_k} \tag{4.2}$$

for a suitable constant $m(\beta) \nearrow +\infty$ as $\beta \searrow 0$.

(b) Let $k_0 = 1$. Then, clearly,

$$P_{k_0} \leqslant 2 \sup_{\sigma} P(\exists C \subset \gamma_{\phi_{t_{k_0}}^{\Lambda_{k_0}}(\sigma)}, |C| > 1) \leqslant 4d(1 - e^{-\beta})(20)^d \ll \frac{1}{2(11)^d} \tag{4.3}$$

if $\beta \ll 1$.

Thus, (4.2), (4.3) prove $P_k \leqslant e^{-mt_k}$ and exponential approach to equilibrium follows.

APPENDIX

In this Appendix we verify the detailed balance condition given by (1.4). We have

$$P(\sigma \to \sigma') = \sum_{\gamma \sim \sigma} P(\sigma \to \gamma)\, P(\gamma \to \sigma') \tag{A.1}$$

where γ is a bond configuration and $P(\sigma \to \gamma) \equiv P(\gamma_\sigma = \gamma)$, $P(\gamma \to \sigma')$ is the conditional probability of $\phi_1^{\Lambda,\omega}(\sigma) = \sigma'$ given $\gamma = \gamma_0$, and $\gamma \sim \sigma$ means that if $\sigma_b = -1$, then b is vacant in γ; in other words, $(1 + \sigma_b)/2 \geqslant \gamma_b$. From the definition of the dynamics it is also clear that

$$P(\gamma \to \sigma') = 0 \qquad \text{unless} \quad \sigma' \sim \gamma$$

Thus, we have

$$\begin{aligned} P(\sigma \to \sigma') &= \sum_{\substack{\gamma \sim \sigma \\ \gamma \sim \sigma'}} P(\sigma \to \gamma)\, P(\gamma \to \sigma') \\ P(\sigma' \to \sigma) &= \sum_{\substack{\gamma \sim \sigma' \\ \gamma \sim \sigma}} P(\sigma' \to \gamma)\, P(\gamma \to \sigma) \end{aligned} \tag{A.2}$$

Now, according to the rule (1.1), we have

$$P(\sigma \to \gamma) = (1 - e^{-\beta})^{\#\{b; \gamma_b = 1\}}\, (e^{-\beta})^{\#\{b; \gamma_b = 0 \text{ and } \sigma_b = 1\}}$$

Thus

$$\begin{aligned} \frac{P(\sigma \to \gamma)}{P(\sigma' \to \gamma)} &= \exp\{-\beta[\#\{b; \sigma'_b = -1\} - \#\{b; \sigma_b = -1\}]\} \\ &= \exp\left[-\frac{\beta}{2}\left(-\sum_b \sigma'_b + \sum_b \sigma_b\right)\right] \end{aligned} \tag{A.3}$$

Similarly,

$$\frac{P(\gamma \to \sigma')}{P(\gamma \to \sigma)} = \prod_{C \subset \gamma C \cup \partial \Lambda = \varnothing} \frac{P(\sigma' \mid C)}{P(\sigma \mid C)} \tag{A.4}$$

with $P(\sigma \mid C) = \{1 + \exp[-\beta h \sum_{x \in C} \sigma(x)]\}^{-1}$.

Since

$$\frac{P(\sigma' \mid C)}{P(\sigma \mid C)} = \exp\left(-\frac{\beta}{2}\left\{-h \sum_{x \in C} [\sigma'(x) - \sigma(x)]\right\}\right)$$

it is easy to check that (A.3), (A.4) give (1.4).

REFERENCES

1. R. H. Swendsen and J. S. Wang, Nonuniversal critical dynamics in Monte Carlo simulations, *Phys. Rev. Lett.* **58** (1987).
2. Xiao-Jian-Li and A. D. Sokal, Rigorous lower bound on the dynamical critical exponent of the Swendsen–Wang dynamics, *Phys. Rev. Lett.* **63**(8):827 (1989).
3. R. G. Edwards and A. D. Sokal, Generalization of the Fortuin–Kasteleyn–Swendsen–Wang representation and Monte Carlo algorithm, *Phys. Rev. D* **38**(6):2009 (1988).
4. P. W. Kasteleyn and C. M. Fortuin, On the random cluster model. I. Introduction and relation to other models, *Physica* **57**:536 (1972).
5. M. Aizenman, J. T. Chayes, L. Chayes, and C. M. Newman, The phase boundary in dilute and random Ising and Potts ferromagnets, *J. Phys. A* **20**:L313 (1987).
6. R. Holley, Possible rates of convergence in finite range, attractive spin systems, *Contemp. Math.* **41** (1985).
7. R. Holley, Rapid convergence to equilibrium in one dimensional stochastic Ising models, *Ann. Prob.* **13**:72–89 (1985).
8. F. Martinelli, E. Oliveri, and E. Scoppola, Metastability and exponential approach to equilibrium for low temperature stochastic Ising models, preprint, Rome (1989).
9. F. Martinelli, E. Oliveri, and E. Scoppola, On the Swendsen–Wang dynamics. II. Critical droplets and homogeneous nucleation at low temperature for the 2 dimensional Ising model, *J. Stat. Phys.* **62**:135–159 (1991).
10. F. Martinelli and E. Scoppola, Small random perturbations of dynamical systems: Exponential loss of memory of the initial condition, *Commun. Math. Phys.* **120**:25–69 (1988).
11. F. Martinelli, E. Oliveri, and E. Scoppola, Small random perturbations of finite and infinite dimensional dynamical system: Unpredictability of exit times, *J. Stat. Phys.* **55**(3/4) (1989).
12. F. Martinelli, E. Oliveri, and E. Scoppola, On the loss of memory of initial conditions for some stochastic flows, in *Proceedings of the 2nd Ascona-Como International Conference, "Stochastic Process-Geometry & Physics,"* S. Albeverio, ed. (World Scientific, Singapore).
13. R. Von Dreyfus, On the effect of randomness in ferromagnetic models and Schrödinger operators, Ph.D. Thesis, New York (1987).
14. A. Klein and R. Von Dreyfus, A new proof of localization in the Anderson tight binding model, *Commun. Math. Phys.* **124**:285–299 (1989).
15. D. A. Huse and D. S. Fisher, *Phys. Rev. B* **35**:6841 (1987).
16. A. D. Sokal and L. E. Thomas, Absence of mass gap for a class of stochastic contour models, *J. Stat. Phys.* **51**:907 (1988).